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Approximation by the Parametric Generalizations of the Modified Bernstein-Durrmeyer Operators

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ABSTRACT

In this article, the Durrmeyer-type generalization of the parametric modified Bernstein operators is studied. Then the central moments and the Korovkin-type theorem are given. Rate of approximation of the constructed operators are found by using modulus of continuity, with the help of functions from Lipschitz class and Peetre- $\mathcal{K}$ functionals.

1. Introduction

Approximation theory has an important place in studies in the field of mathematics. In 1912, Bernstein (1912) proved the approximation theorem defined by Weierstrass (1885) on the closed interval $[0,1]$. In 2017, Chen et al. (2017) gave some approximation properties of Bernstein polynomials, called the θ −Bernstein polynomials. In 2018, the original Bernstein–Durrmeyer type operators were defined and studied by Acar et al. (2018). Based on these operators, Usta (2020) defined the modified Bernstein operators for $g \in C[0,1], \eta \in N$ and $x \in (0,1)$ in 2020 as

$$B_\eta^*(g;x) = \frac{1}{\eta}\sum_{\zeta=0}^{\eta}\binom{\eta}{\zeta}(\zeta-\eta x)^2 x^{\zeta-1}(1-x)^{\eta-\zeta-1} g\left(\frac{\zeta}{\eta}\right).$$

He evidenced the $B_\eta^*(g;x)$ operators as follows

$$B_\eta^*(1;x) = 1$$

$$B_\eta^*(t;x) = \frac{\eta-2}{\eta}x + \frac{1}{\eta}$$

$$B_\eta^*(t^2;x) = \frac{\eta^2-7\eta+6}{\eta^2}x^2 + \frac{5\eta-6}{\eta^2}x + \frac{1}{\eta^2}.$$

References (Cai et al., 2018; Aral and Erbay, 2019; Srivastava et al., 2019; Kajla et al., 2020; Mohiuddine and Özger, 2020; Kadak and Özger, 2021; Mohiuddine et al., 2021) are examples of articles on parametric generalizations.

After that, θ parameterization of modified Bernstein operators defined for every $g \in C[0,1]$ by Sofyalıoğlu et al. (2020) as

$$B_{\eta,\theta}^*(g;x) = \sum_{\zeta=0}^{\eta}\rho_{\eta,\zeta}^{(\theta)}(x) g\left(\frac{\zeta}{\eta}\right), \qquad (1)$$

where $\eta \geq 2$, $0 \leq \theta \leq 1$, $x \in (0,1)$ and

$$\rho_{1,0}^{(\theta)}(x) = x, \qquad \rho_{1,1}^{(\theta)}(x) = 1-x,$$

$$\rho_{\eta,\zeta}^{(\theta)}(x) = \left\{\frac{1}{\eta-1}\binom{\eta-2}{\zeta}(\zeta-(\eta-1)x)^2(1-\theta)x + \frac{1}{\eta-1}\binom{\eta-2}{\zeta-2}\left((\zeta-1)-(\eta-1)x\right)^2(1-\theta)(1-x) + \frac{1}{\eta}\binom{\eta}{\zeta}(\zeta-\eta x)^2\theta x(1-x)\right\}x^{\zeta-2}(1-x)^{\eta-\zeta-2}$$

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with binomial coefficients

$$\binom{\eta}{\zeta}=\begin{cases}\dfrac{\eta!}{(\eta-\zeta)!\,\zeta!} & \text{if } 0\le\zeta\le\eta\\ 0 & \text{otherwise}\end{cases}.$$

They obtained the $B^*_{\eta,\theta}(g;x)$ operators as follows

$$B^*_{\eta,\theta}(1;x)=1$$

$$B^*_{\eta,\theta}(t;x)=\frac{\eta-2}{\eta}x+\frac{1}{\eta}$$

$$B^*_{\eta,\theta}(t^2;x)=\frac{(\eta+1-2\theta)(\eta^2-7\eta+6)}{\eta^2(\eta-1)}x^2+\frac{(5\eta^2-11\theta\eta+14\theta-8)}{\eta^2(\eta-1)}x+\frac{\eta+2-3\theta}{\eta^2(\eta-1)}.$$

Selectioning $\theta=1$, the operators $B^*_{\eta,\theta}(g;x)$ are reduced to $B^*_{\eta}(g;x)$ given by Usta (2020). Now, we introduce the parametric generalization of the modified Bernstein-Durrmeyer operators

$$D^*_{\eta,\theta}(g;x)=\frac{\eta+1}{\eta}\sum_{\zeta=0}^{\eta}\rho^{(\theta)}_{\eta,\zeta}(x)\int_0^1\rho_{\eta,\zeta}(t)g(t)dt,$$

where $\eta\ge 2$, $0\le\theta\le 1$, $x\in(0,1)$ and

$$\rho^{(\theta)}_{1,0}(x)=x,\qquad \rho^{(\theta)}_{1,1}(x)=1-x,$$

$$\rho^{(\theta)}_{\eta,\zeta}(x)=\left\{\frac{\eta}{\eta-1}\binom{\eta-2}{\zeta}(\zeta-(\eta-1)x)^2(1-\theta)x+\frac{\eta}{\eta-1}\binom{\eta-2}{\zeta-2}(\zeta-1-(\eta-1)x)^2(1-\theta)(1-x)+\binom{\eta}{\zeta}(\zeta-\eta x)^2\theta x(1-x)\right\}x^{\zeta-2}(1-x)^{\eta-\zeta-2}$$

and

$$\rho_{\eta,\zeta}(t)=\binom{\eta}{\zeta}(\zeta-\eta t)^2t^{\zeta-1}(1-t)^{\eta-\zeta-1}.$$

2. Auxiliary results

Lemma 2.1

For every $x\in(0,1)$, the operator $D^*_{\eta,\theta}(t;x)$ has the following identities:

$$D^*_{\eta,\theta}(1;x)=1,$$

$$D^*_{\eta,\theta}(t;x)=\frac{(\eta-2)^2}{\eta(\eta+2)}x+\frac{3\eta-2}{\eta(\eta+2)},$$

$$D^*_{\eta,\theta}(t^2;x)=\frac{[(\eta^4-12\eta^3+35\eta^2+12\eta-36)-\theta(2\eta^3-26\eta^2+96\eta-72)]}{\eta(\eta+2)(\eta+3)(\eta-1)}x^2+\frac{[(57\eta^2-24\eta-36)-\theta(11\eta^2+80\eta+84)]}{\eta(\eta+2)(\eta+3)(\eta-1)}x+\frac{[(14\eta^2-23\eta-6)-\theta(3\eta-18)]}{\eta(\eta+2)(\eta+3)(\eta-1)}.$$

Proof. For $g(t)=1$, we get

$$D^*_{\eta,\theta}(1;x)=\frac{\eta+1}{\eta}\sum_{\zeta=0}^{\eta}\rho^{(\theta)}_{\eta,\zeta}(x)\int_0^1\binom{\eta}{\zeta}(\zeta-\eta t)^2t^{\zeta-1}(1-t)^{\eta-\zeta-1}dt$$

$$=\frac{\eta+1}{\eta}B^*_{\eta,\theta}(1;x)\frac{\eta}{\eta+1}$$

$$=1.$$

For $g(t)=t$, we have

$$D^*_{\eta,\theta}(t;x)=\frac{\eta+1}{\eta}\sum_{\zeta=0}^{\eta}\rho^{(\theta)}_{\eta,\zeta}(x)\int_0^1\binom{\eta}{\zeta}(\zeta-\eta t)^2t^{\zeta}(1-t)^{\eta-\zeta-1}dt$$

$$=\frac{\eta+1}{\eta}\sum_{\zeta=0}^{\eta}\rho^{(\theta)}_{\eta,\zeta}(x)\frac{(\eta-2)\zeta+2\eta}{(\eta+1)(\eta+2)}$$

$$=\frac{\eta-2}{\eta+2}B^*_{\eta,\theta}(t;x)+\frac{2}{\eta+2}$$

$$=\frac{(\eta-2)^2}{\eta(\eta+2)}x+\frac{3\eta-2}{\eta(\eta+2)}.$$

For $g(t)=t^2$, we obtain

$$D^*_{\eta,\theta}(t^2;x)=\frac{\eta+1}{\eta}\sum_{\zeta=0}^{\eta}\rho^{(\theta)}_{\eta,\zeta}(x)\int_0^1\binom{\eta}{\zeta}(\zeta-\eta t)^2t^{\zeta+1}(1-t)^{\eta-\zeta-1}dt$$

$$=\frac{\eta+1}{\eta}\sum_{\zeta=0}^{\eta}\rho^{(\theta)}_{\eta,\zeta}(x)\frac{\zeta^2(\eta-6)+\zeta(7\eta-6)+6\eta}{(\eta+1)(\eta+2)(\eta+3)}$$

$$=\frac{\eta(\eta-6)}{(\eta+2)(\eta+3)}B^*_{\eta,\theta}(t^2;x)\frac{7\eta-6}{(\eta+2)(\eta+3)}B^*_{\eta,\theta}(t;x)+\frac{6}{(\eta+2)(\eta+3)}$$

$$=\frac{[(\eta^4-12\eta^3+35\eta^2+12\eta-36)-\theta(2\eta^3-26\eta^2+96\eta-72)]}{\eta(\eta+2)(\eta+3)(\eta-1)}x^2$$

$$+\frac{[(57\eta^2-24\eta-36)-\theta(11\eta^2+80\eta+84)]}{\eta(\eta+2)(\eta+3)(\eta-1)}x+\frac{[(14\eta^2-23\eta-6)-\theta(3\eta-18)]}{\eta(\eta+2)(\eta+3)(\eta-1)}.$$

Lemma 2.2. For every $x\in(0,1)$, we get the central moments as

$$D^*_{\eta,\theta}(t-x;x)=\frac{(-6\eta+4)x+(3\eta-2)}{\eta(\eta+2)}$$

$$D^*_{\eta,\theta}((t-x)^2;x)=\frac{1}{\eta(\eta+2)(\eta+3)(\eta-1)}\{-12x^2-48x+\eta^3(-4x^2-6x+3)$$

$$+\eta^2(50x^2+49x+18)+\eta(-34x^2+2x-36)$$

$$-\theta[(2\eta^3-26\eta^2+96\eta-72)x^2+(11\eta^2+80\eta+84)x+(3\eta-18)]\}.$$

Proof. Briefly, central moments can express as

$$D^*_{\eta,\theta}(t-x;x)=D^*_{\eta,\theta}(t;x)-xD^*_{\eta,\theta}(1;x),$$

$$D^*_{\eta,\theta}((t-x)^2;x)=D^*_{\eta,\theta}(t^2;x)-2xD^*_{\eta,\theta}(t;x)+x^2D^*_{\eta,\theta}(1;x).$$

We can find the result of the lemma.

Let $C[0,1]$ be the Banach space of all continuous functions g on $[0,1]$ with the norm

$$\|g\| = \sup_{x\in(0,1)} |g(x)|.$$

Theorem 2.1. For every $x \in (0,1)$ and $g \in C[0,1]$

$$\left\|D^*_{\eta,\theta}(g;x) - g(x)\right\| \rightrightarrows 0. \quad (3)$$

Proof. In the light of Korovkin theorem (1953) and $e_i(t) = t^i, i = 0,1,2$ the proof is completed with the

$$\lim_{\eta\to\infty} D^*_{\eta,\theta}(e_i;x) = x^i, \qquad i = 0,1,2$$

equations.

3. Rate of Convergence

The modulus of continuity is stated by

$$\omega(g,\delta) := \sup_{|t-x|\le\delta} \sup_{x\in(0,1)} |g(t) - g(x)|, \qquad \delta > 0,$$

where $g \in C[0,1]$.

It is due to the following feature of the modulus of continuity

$$|g(t) - g(x)| \le \left(1 + \frac{|t-x|}{\delta}\right)\omega(g,\delta).$$

Theorem 3.1. For every $x \in (0,1)$ and $g \in C[0,1]$,

$$\left|D^*_{\eta,\theta}(g;x) - g(x)\right| \le 2\omega(g,\delta_\eta). \quad (4)$$

Here,

$$\delta_\eta(x) = \left[D^*_{\eta,\theta}((t-x)^2;x)\right]^{\frac{1}{2}}.$$

Proof. For $D^*_{\eta,\theta}$,

$$\begin{aligned}\left|D^*_{\eta,\theta}(g;x) - g(x)\right| &= \left|D^*_{\eta,\theta}(g(t) - g(x);x)\right| \\ &\le D^*_{\eta,\theta}(|g(t) - g(x)|;x) \\ &\le \omega(g,\delta)\left\{D^*_{\eta,\theta}(1;x) + \frac{1}{\delta}D^*_{\eta,\theta}(|t-x|;x)\right\} \\ &\le \omega(g,\delta)\left\{1 + \frac{1}{\delta}\left[D^*_{\eta,\theta}((t-x)^2;x)\right]^{\frac{1}{2}}\right\}.\end{aligned}$$

If we choose

$$\delta = \delta_\eta = \left[D^*_{\eta,\theta}((t-x)^2;x)\right]^{\frac{1}{2}},$$

then we get

$$|D^*_{\eta,\theta}(g;x) - g(x)| \le 2\omega\left(g, \left[D^*_{\eta,\theta}((t-x)^2;x)\right]^{\frac{1}{2}}\right),$$

which completes the proof.

Here, we estimate the rate of convergence of $D^*_{\eta,\theta}(g;x)$ by using functions of Lipschitz class.

Let's recall that a function $g \in Lip_M(\varsigma)$ on $(0,1)$ if the inequality

$$|g(t) - g(x)| \le M|t-x|^\varsigma \ ; \ \forall\, t,x \in (0,1) \quad (5)$$

holds.

Theorem 3.2. Let $x \in (0,1)$, $g \in Lip_M(\varsigma)$, $0 < \varsigma \le 1$, then we get

$$\left|D^*_{\eta,\theta}(g;x) - g(x)\right| \le M\delta^\varsigma_\eta(x),$$

where $\delta_\eta(x) = \left[D^*_{\eta,\theta}((t-x)^2;x)\right]^{\frac{1}{2}}$.

Proof. Let $x \in (0,1)$, $g \in Lip_M(\varsigma)$ and $0 < \varsigma \le 1$. From the linearity and monotonicity of the operators $D^*_{\eta,\theta}$ we have

$$\begin{aligned}\left|D^*_{\eta,\theta}(g;x) - g(x)\right| &\le D^*_{\eta,\theta}(|g(t) - g(x)|;x) \\ &\le MD^*_{\eta,\theta}(|t-x|^\varsigma;x).\end{aligned}$$

By putting $p = \frac{2}{\varsigma}$, $q = \frac{2}{2-\varsigma}$ in the Hölder inequality, we obtain

$$\left|D^*_{\eta,\theta}(g;x) - g(x)\right| \le M\left[D^*_{\eta,\theta}((t-x)^2;x)\right]^{\frac{\varsigma}{2}} \le M\delta^\varsigma_\eta(x).$$

By selecting

$$\delta_\eta(x) = \left[D^*_{\eta,\theta}((t-x)^2;x)\right]^{\frac{1}{2}}$$

we get the result.

Ultimately, we mention the rate of convergence of $D^*_{\eta,\theta}(g;x)$ operators with the help of Peetre$-\mathcal{K}$ functionals. Firstly, we offer the following auxiliary lemma, which will be used in the proof of the main theorem. For $x \in (0,1)$ and $g \in C[0,1]$, we get

$$\left|D^*_{\eta,\theta}(g;x)\right| \leq \|g\|. \tag{6}$$

Theorem 3.3. Let $x \in (0,1)$ and $g \in C[0,1]$. For every $\eta \in \mathbb{N}$ we write

$$|D^*_{\eta,\theta}(g;x) - g(x)| \leq 2\mathcal{K}(g;\lambda_\eta(x)),$$

where

$$\lambda_\eta(x) = \frac{2}{4\eta(\eta+2)(\eta+3)(\eta-1)}\big|(9\eta^3+26\eta^2-62\eta+12-3\theta(\eta-6))$$
$$+\big(-18\eta^3+33\eta^2+54\eta-72-\theta(11\eta^2+80\eta+84)\big)x$$
$$+\big(-4\eta^3+50\eta^2-34\eta-12-\theta(2\eta^3-26\eta^2+96\eta-72)\big)x^2\big|.$$

Proof. For a given function $h \in C^2[0,1]$, we have the following Taylor polynomial

$$h(t) = h(x) + (t-x)h'(x) + \int_x^t (t-s)h''(s)ds, \qquad t \in (0,1). \tag{7}$$

Applying $D^*_{\eta,\theta}$ operator to both sides of the equation (7), we get

$$\left|D^*_{\eta,\theta}(h;x) - h(x)\right| = \left|D^*_{\eta,\theta}((t-x)h'(x);x)\right| + \left|D^*_{\eta,\theta}\left(\int_x^t (t-s)h''(s)ds;x\right)\right|$$
$$\leq \|h'\|\left|D^*_{\eta,\theta}(t-x;x)\right| + \|h''\|\left|D^*_{\eta,\theta}\left(\int_x^t (t-s)ds;x\right)\right|$$
$$\leq \|h'\|\left|D^*_{\eta,\theta}(t-x;x)\right| + \|h''\|\frac{1}{2}D^*_{\eta,\theta}((t-x)^2;x).$$

So,

$$\left|D^*_{\eta,\theta}(h;x) - h(x)\right| \leq \lambda\|h\|.$$

Using the above inequality, we obtain

$$\left|D^*_{\eta,\theta}(g;x) - g(x)\right| = \left|D^*_{\eta,\theta}(g;x) - g(x) + D^*_{\eta,\theta}(h;x) - D^*_{\eta,\theta}(h;x) + h(x) - h(x)\right|$$
$$\leq \|g-h\|\left|D^*_{\eta,\theta}(1;x)\right| + \|g-h\| + \left|D^*_{\eta,\theta}(h;x) - h(x)\right|$$
$$\leq 2(\|g-h\| + \lambda\|h\|)$$
$$= 2\mathcal{K}(g;\lambda_\eta). \tag{8}$$

$$\left|D^*_{\eta,\theta}(g;x) - g(x)\right| \leq 2\mathcal{K}\left(g;\lambda_\eta(x)\right). \tag{9}$$

So the proof is finished.

Declaration of Competing Interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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