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Analytically Evaluation of Bose-Einstein Type Integral by the Binomial Series Expansions

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ABSTRACT

The Bose-Einstein integral functions are important because they arise in various numerical calculations of different domains of physics. In this study, using the Binomial expansion theorem an analytical formula was developed that allows the Bose-Einstein integral for states $\eta < 0$ to be calculated without resorting to numerical integration of whole and half integer integrals. The results obtained were compared with the results of the literature and it was seen that they were in agreement.

1. Introduction

As is known, the Bose-Einstein function arises as a quantum statistical distribution. However, besides quantum statistics, it also comes with many different forms of various physical problems. For example, the Bose–Einstein (BE) function describes the specific energy, other physical quantities of the boson, such as photons, phonons, magnons, Cooper pair fermions, or atoms with integer rotations. So, it is an important special function that is often used in solid state physics (Kittel, 2006), statistical mechanics (Landau and Lifschitz, 1994; Tassaddiq, 2022) and astrophysics (Weiss et al., 2006). Therefore, it is very useful to develop by applying them to various physical and engineering problems of mathematics with study such integrals/functions as analytical functions. For this reason, it has been extensively studied by many scientists (Dinge, 1957; Pichon, 1989; Natarajan and Mohankumar, 1993; Gong et al., 2001; Bhagat et al., 2003; Chaudhry and Qadir, 2007; Lee, 1997; Bordag, 2020).The complete form of the Bose-Einstein integral is written as follows (Bhagat et al., 2003; Chaudhry and Qadir, 2007; Lee, 1997):

$$B_p(x) = \frac{1}{\Gamma(p+1)} \int_0^\infty \frac{y^p}{e^{y-\eta} - 1} dy$$

Here, $\Gamma(n)$ is the gamma function of the n argument. The normalization multiplier is defined for cases where the integral is positive and $\eta \leq 0$ and $0 < p$ or $\eta < 0$ and $-1 < p \leq 0$, except for $\Gamma(p+1)$ case. On the other hand, when $\eta > 0$, the integral can be defined mathematically (Pichon, 1989). However, such cases are not physical, as they contain imaginary roots. There are studies in the literature that discuss the algebraic state of the integral appropriately by using serial expansions for the solution of such integrals (Clunie, 1954; Wolfram, 2003; Cvijovi, 2007). In the study by Tassaddiq and Qadir, Fourier transform representations are represented using Fermi-Dirac and Bose – Einstein functions, which are extended with applications to the zeta and related functions (Tassaddiq and Qadir, 2011). In the study, a new series representation is constructed for generalized Bose-Einstein and Fermi-Dirac functions using the fractional Weyl transform (Srivastava, et al., 2019). In another the study conducted by Tassaddiq (2022), a serial representation of the Bose-Einstein and Fermi-Dirac functions in terms of delta functional of the complex argument affecting the Z space of the test functions was obtained. In addition, this integral has been extensively investigated numerically (Gautschi, 1993; Odrzywolek, 2011) and

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analytically (Gradshteyn and Ryzhik, 1980; Chaudhry and Qadir, 2007; Selvaggi and Selvaggi, 2018)

However, in practical applications not only the definitions of integer states, but also some fractional coefficient integrals are needed. For example, the number density of the boson gas distributed in the one-dimensional bar, two-dimensional plate and three-dimensional box is defined by $1/2\,, 0, -1/2$ coefficient integrals, respectively.

In addition, such integrals often occur in the mathematical modeling of semiconductor devices. However, their evaluation in the numerical schemes used to solve these equations can be computationally difficult. Therefore, it is necessary to be able to calculate these integrals quickly and with sufficient accuracy using appropriate approaches. In this study, we propose an analytical statement for the Bose-Einstein integral using the Binomial series expansions (Eser et al., 2011; 2020; Eser and Koç, 2021; Koç and Eser, 2016; 2019), which greatly saves labor.

2. Theoretical Method

The Bose–Einstein function is defined by its integral notation as follows:

$$B_p(x) = \frac{1}{\Gamma(p+1)} \int_0^\infty \frac{(t+\eta)^p}{e^t - 1} dt \tag{1}$$

or

$$B_p(x) = \frac{1}{\Gamma(p+1)} \int_0^\infty \frac{t^p}{e^{t-\eta} - 1} dt \tag{2}$$

Where η varies in the range $-\infty < \eta < +\infty$, and the parameter takes optional integer and non-integer values. The following equation can be obtained by using the binomial series expansion form for the function $(e^{t-\eta} - 1)^{-1}$ in Eq. (2) and making the necessary simplifications:

$$(x \pm y)^n = \begin{cases} \sum_{m=0}^{\infty} (\pm 1)^m f_m(n) x^{n-m} y^m, & n \neq integer \\ \sum_{m=0}^{n} (\pm 1)^m f_m(n) x^{n-m} y^m, & n = integer \end{cases} \tag{3}$$

Accordingly;

$$(e^{t-\eta} - 1)^{-1} = \lim_{N\to\infty} \sum_{m=1}^{N} (\pm 1)^m f_m(n) \begin{cases} e^{m(\eta - t)}, & for \quad t > \eta \\ -1 - e^{-m(\eta - t)}, & for \quad t < \eta \end{cases} \tag{4}$$

Now, for $t > \eta$, considering Eq. (3) in Eq.(2);

$$B_p(x) = \frac{1}{\Gamma(p+1)} \lim_{N\to\infty} \sum_{m=0}^{N} (-1)^m f_m(-1) \int_0^\infty t^p e^{-m(\eta - t)} dt \tag{5}$$

We get the following formula for the Bose – Einstein function:

$$B_p(x) = \frac{1}{\Gamma(p+1)} \lim_{N\to\infty} \sum_{m=1}^{N} (-1)^m f_m(-1)\, e^{m\eta} \frac{\Gamma(p+1)}{m^{p+1}}$$

$$= \lim_{N\to\infty} \sum_{m=1}^{N} (-1)^m f_m(-1)\, e^{m\eta} m^{-(p+1)} \tag{6}$$

In Eqs. (6), the N are upper limit of summation. The quantities $\Gamma(n)$, in Eq. (5) are well known familiar functions defined by (Gradshteyn and Ryzhik, 1980):

$$\Gamma(n) = \int_0^\infty t^{n-1} e^{-t} dt \tag{7}$$

3. Results and Discussion

Accurate evaluation of the generalized Bose-Einstein integral and its derivatives are very important for various fields of physics and astrophysics. The results of these integrals are usually obtained through numerical integration. In this study, a simple analytical expression using binomial functions is proposed for ease of calculation. Using the technique of binomial function expansion for such integrals is one of the most effective methods for solving equations.

In this study, the calculation results for the integer and half-integer states (-9/2....50/2) of the Bose-Einstein integral with the proposed analytic expression (Eq. 6) are presented in Table 1. As can be seen from the results, the results are in harmony with the literature values. The results were determined with the help of the Mathematica 12 program created for the analytical expression of Eq. (6). The results of the analysis of this study can be a useful guide for advanced theoretical and experimental studies of many physical systems.

Table 1. Values used in the new analytical formulae to compute the BE integrals of some positive /negative integer orders.

2p		***η = -1.5***	***η = -1.0***	***η = -0.75***
-9	a	1.8813725967014886	11.637629665446079	42.454945144395927
	b	1.8813725967014896	11.637629675127071	42.454973112334442
-7	a	0.8042111607394022	3.3264089046605520	9.1007698618167616
	b	0.8042111607394021	3.3264089049663461	9.1007707348113005
-5	a	0.4501997635247233	1.2979450676568245	2.6984249681102447
	b	0.4501997635247233	1.2979450676664934	2.6984249954112056
-3	a	0.3193731919000945	0.7072407184864977	1.1777115203552861
	b	0.3193731919000946	0.7072407184868038	1.1777115212105438
-1	a	0.2662989807334175	0.5060301198729264	0.7345109543941865
	b	0.2662989807334175	0.5060301198729361	0.7345109544210222
1	a	0.2432400058683661	0.4284407345998377	0.5810878479900716
	b	0.2432400058683661	0.4284407345998379	0.5810878479909148
2	a	0.2369924122759423	0.4087542873488962	0.5443905122616534
	b	0.2369924122759424	0.4087542873488963	0.5443905122618031
3	a	0.2327330035613081	0.3957280103803375	0.5207364456600071
	b	0.2327330035613081	0.3957280103803376	0.5207364456600337
4	a	0.2298088090601440	0.3869954242101997	0.5052016032356575
	b	0.2298088090601440	0.3869954242101997	0.5052016032356623
5	a	0.2277899169462501	0.3810793119677888	0.4948448400827057
	b	0.2277899169462502	0.3810793119677889	0.4948448400827065
6	a	0.2263896760250073	0.3770372769243202	0.4878571450793349
	b	0.2263896760250073	0.3770372769243203	0.4878571450793351
7	a	0.2254149218864423	0.3742568990347962	0.4830975610829914
	b	0.2254149218864423	0.3742568990347961	0.4830975610829913
8	a	0.2247343374367560	0.3723339822804351	0.4798311019792835
	b	0.2247343374367561	0.3723339822804351	0.4798311019792836
9	a	0.2242580001431964	0.3709983097129550	0.4775759128698879
	b	0.2242580001431964	0.3709983097129550	0.4775759128698880
10	a	0.2239239652370866	0.3700673152590212	0.4760114973557493
	b	0.2239239652370866	0.3700673152590211	0.4760114973557492
11	a	0.2236893523817355	0.3694165828905786	0.4749221586519055
	b	0.2236893523817355	0.3694165828905787	0.4749221586519055
12	a	0.2235243600275573	0.3689607273747987	0.4741613379802982
	b	0.2235243600275574	0.3689607273747988	0.4741613379802982
13	a	0.2234082091329660	0.3686408150523765	0.4736286820530093
	b	0.2234082091329661	0.3686408150523767	0.4736286820530094
14	a	0.2233263733880340	0.3684159813379485	0.4732550470373619
	b	0.2233263733880341	0.3684159813379485	0.4732550470373619
15	a	0.2232686759337352	0.3682577848601628	0.4729925532832373
	b	0.2232686759337353	0.3682577848601629	0.4729925532832372
16	a	0.2232279746636775	0.3681463709315172	0.4728079118886616
	b	0.2232279746636775	0.3681463709315173	0.4728079118886615
17	a	0.2231992501856333	0.3680678453540935	0.4726779031186648
	b	0.2231992501856333	0.3680678453540937	0.4726779031186649
18	a	0.2231789708869173	0.3680124658840481	0.4725862883270347
	b	0.2231789708869174	0.3680124658840481	0.4725862883270348
19	a	0.2231646496363309	0.3679733906334758	0.4725216871342709
	b	0.2231646496363310	0.3679733906334759	0.4725216871342708
20	a	0.2231545335536795	0.3679458084142871	0.4724761103816445
	b	0.2231545335536795	0.3679458084142871	0.4724761103816446
21	a	0.2231473864860059	0.3679263324884449	0.4724439418951736
	b	0.2231473864860059	0.3679263324884449	0.4724439418951737
22	a	0.2231423362480243	0.3679125768157347	0.4724212292743981
	b	0.2231423362480242	0.3679125768157348	0.4724212292743982

23	a	0.2231387672074460	0.3679028592205868	0.4724051885129940
	b	0.2231387672074460	0.3679028592205869	0.4724051885129941
24	a	0.2231362446766686	0.3678959930981207	0.4723938571805314
	b	0.2231362446766686	0.3678959930981207	0.4723938571805313
25	a	0.2231344616477140	0.3678911410410195	0.4723858511583397
	b	0.2231344616477141	0.3678911410410196	0.4723858511583397
26	a	0.2231332012418684	0.3678877118602080	0.4723801937510376
	b	0.2231332012418684	0.3678877118602081	0.4723801937510375
27	a	0.2231323102229115	0.3678852880664591	0.4723761954928840
	b	0.2231323102229115	0.3678852880664592	0.4723761954928841
28	a	0.2231316803057103	0.3678835747635519	0.4723733695257255
	b	0.2231316803057104	0.3678835747635520	0.4723733695257256
29	a	0.2231312349610102	0.3678823636086397	0.4723713719727784
	b	0.2231312349610103	0.3678823636086397	0.4723713719727785
30	a	0.2231309200974532	0.3678815073848735	0.4723699598972964
	b	0.2231309200974532	0.3678815073848736	0.4723699598972965
31	a	0.2231306974799311	0.3678809020540019	0.4723689616441773
	b	0.2231306974799311	0.3678809020540020	0.4723689616441772
32	a	0.2231305400797847	0.3678804740843146	0.4723682559081293
	b	0.2231305400797847	0.3678804740843146	0.4723682559081292
33	a	0.2231304287892751	0.3678801715008908	0.4723677569555202
	b	0.2231304287892752	0.3678801715008909	0.4723677569555201
34	a	0.2231303500997338	0.3678799575633546	0.4723674041878111
	b	0.2231303500997338	0.3678799575633547	0.4723674041878110
35	a	0.2231302944605568	0.3678798062989347	0.4723671547693658
	b	0.2231302944605568	0.3678798062989347	0.4723671547693657
36	a	0.2231302551192937	0.3678796993459131	0.4723669784188875
	b	0.2231302551192937	0.3678796993459132	0.4723669784188876
37	a	0.2231302273017296	0.3678796237227891	0.4723668537289204
	b	0.2231302273017296	0.3678796237227892	0.4723668537289205
38	a	0.2231302076322665	0.3678795702515215	0.4723667655647914
	b	0.2231302076322665	0.3678795702515216	0.4723667655647913
39	a	0.2231301937241588	0.3678795324429854	0.4723667032262186
	b	0.2231301937241588	0.3678795324429854	0.4723667032262187
40		0.2231301838898165	0.3678795057090979	0.4723666591478536
41		0.2231301769359874	0.3678794868058378	0.4723666279807022
42		0.2231301720189460	0.3678794734394758	0.4723666059427519
43		0.2231301685421063	0.3678794639881815	0.4723665903598874
44		0.2231301660836289	0.3678794573051943	0.4723665793413228
45		0.2231301643452340	0.3678794525796591	0.4723665715501275
46		0.2231301631160096	0.3678794492382301	0.4723665660409820
47		0.2231301622468205	0.3678794468754998	0.4723665621454633
48		0.2231301616322132	0.3678794452048068	0.4723665593909361
49		0.2231301611976214	0.3678794440234540	0.4723665574432030
50		0.2231301608903193	0.3678794431881147	0.4723665560659546

a is this study
b is taken from (Fukushima, 2020)

4. Conclusion

This paper is mainly concerned with the numerical evaluation of the Bose-Einstein integral function. The present work follows the discussion of Bose-Einstein integral. It is found that the Bose-Einstein integrals of positive as well as negative integer can be computed to a very high order of accuracy using the method obtained the present study.
Consequently, we can say with the assistance of the integral of the Bose-Einstein resolved in this study, many physical problems can be resolved, such as particle density and relaxation time.

Declaration of Competing Interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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