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Midset Structure and Equilateral Rigidity in Ultrametric Spaces

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ABSTRACT

For distinct points x, y in an ultrametric space (X, d), we prove that the midset $M(x,y)=\{z: d(x,z)=d(y,z)\}$ equals $X \setminus (B(x,r) \cup B(y,r))$ where $r=d(x,y)$, hence is clopen. This yields a dendrogram formula $|M(x,y)|=n-\lambda(c_i)-\lambda(c_j)$ via the representing tree. Our main results are: (1) an equilateral rigidity theorem-constant midset cardinality forces the space to be equilateral; (2) a subset S is midset-convex if and only if it meets at least three children of its least common ancestor; (3) the Haar measure formula $\mu(M(x,y))=1-2p^{-(k+1)}$ in $\mathbb{Z}_p$ for $d(x,y)=p^{-k}$. We also characterize the Non-Empty Midset Property completely and recover the result that ultrametric spaces with the Unique Midpoint Property have at most three points.

1. Introduction

The equidistant set, or midset, of two distinct points x and y in a metric space (X, d) is the set

$$M(x,y)=\{z \in X: d(x,z)=d(y,z)\}$$

This classical object appears already in the early work of Nadler [6], who studied metric spaces in which the midset is always non-empty and connected, establishing embedding theorems linking such spaces to the real line. Hattori and Ohta [3] proved that a separable metrizable space admits a metric with the Unique Midpoint Property (UMP)-meaning $|M(x,y)|=1$ for every pair-if and only if it is homeomorphic to a subspace of $\mathbb{R}$. Kitai [4] surveys these and related results. Berger [1] places midsets in the broader context of metric geometry. More recently, Dovgoshey and Shcherbak [2] investigated midsets in ultrametric spaces in connection with pretangent spaces, and Vural [10] proved that any ultrametric space with the UMP consists of at most three equidistant points.

Ultrametric spaces-those satisfying the strong triangle inequality

$$d(x,z) \leq \max\{d(x,y), d(y,z)\}$$

-arise in p-adic number theory [7, 9], phylogenetic reconstruction [8], and classification in data science. Their geometric structure is remarkably rigid: every point of an open ball is its center, any two balls are either disjoint or comparable, and every triangle is isosceles [7]. Topologically, ultrametric spaces are zero-dimensional-they admit a base of clopen sets-and compact ultrametric spaces are homeomorphic to closed subspaces of the Cantor set [5]. The ball structure organizes into a rooted tree (the representing tree or dendrogram), a correspondence central to both theory [5] and applications [8].

In this paper we show that midsets in ultrametric spaces are governed by a single clean principle: The midset is the complement of the union of two open balls of the same radius.

Precisely, $M(x,y)=X \setminus (B(x,r) \cup B(y,r))$ where $r=d(x,y)$ (Theorem 3.7). From this we derive:

(1) NMP characterization. A finite ultrametric space has the Non-Empty Midset Property if and only if the root of its representing tree has at least three children (Theorem 4.1), and only the root can obstruct NMP (Proposition 7.5).
(2) Equilateral rigidity. Constant midset cardinality forces a finite ultrametric space to be equilateral (Theorem 7.6).

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(3) Midset-convexity. A subset S with $|S| \geq 2$ is midset-convex if and only if it intersects at least three children of its least common ancestor in the representing tree (Theorem 8.4).
(4) Haar measure of midsets. In $\mathbb{Z}_p (p \geq 3)$, the Haar measure of $M(x,y)$ is $1 - 2p^{-(k+1)}$ when $d(x,y) = p^{-k}$ (Theorem 5.3), with the decomposition into equilateral and far components given explicitly.
(5) UMP recovery. Any ultrametric space with the UMP has at most three points (Corollary 6.3).

Applications. The structural results developed here are relevant to hierarchical clustering algorithms (which often employ ultrametric distance functions), to the study of p-adic geometry and non-Archimedean analysis, and to tree-based models in phylogenetic reconstruction and data science.

2. Preliminaries

Throughout, (X,d) denotes an ultrametric space. We write $B(x,r) = \{z: d(x,z) < r\}$ and $\bar{B}(x,r) = \{z: d(x,z) \leq r\}$. We collect standard properties (see [7, 9]).

Proposition 2.1. Let (X,d) be an ultrametric space.

(i) (Isosceles inequality) If $d(x,y) \neq d(y,z)$, then $d(x,z) = \max\{d(x,y), d(y,z)\}$.

(ii) (Center property) If $y \in B(x,r)$, then $B(x,r) = B(y,r)$; likewise for $\bar{B}$.

(iii) (Nesting) Any two balls are either disjoint or comparable.

(iv) (Clopen structure) Every open ball is closed; every closed ball is open.

(v) (Partition) For any $r > 0, \bar{B}(x,r) = \sqcup_\alpha B(x_\alpha, r)$, a disjoint union of maximal open balls of radius r.

Property (iv) is a topological manifestation of the strong triangle inequality and ensures that ultrametric spaces are zero-dimensional. This will be essential: our midset characterization expresses the midset as a complement of clopen sets, hence clopen itself.

Definition 2.2. The representing tree $\mathcal{T}(X,d)$ of a finite ultrametric space is the rooted tree whose leaves are the points of X, whose internal nodes correspond to the equivalence classes of $x \sim_r y \Leftrightarrow d(x,y) \leq r$ for $r \in \mathcal{D} = \{d(x,y): x \neq y\}$ that are not already classes at a smaller r, and whose root ρ is X itself at height $\operatorname{diam}(X)$. For any node v, we write $\lambda(v)$ for the number of leaf descendants. The least common ancestor lca (x,y) is the node at height $d(x,y)$, and the children of lca (x,y) correspond to the maximal open balls of radius $d(x,y)$ within $\bar{B}(x, d(x,y))$.

3. Midset characterization

Fix distinct $x, y \in X$ and $r = d(x,y)$.

Proposition 3.1 (Lower bound). If $z \in M(x,y)$, then $d(x,z) = d(y,z) \geq r$.
Proof. Setting $s = d(x,z) = d(y,z)$, the strong triangle inequality gives

$$r = d(x,y) \leq \max\{d(x,z), d(z,y)\} = s$$

This partitions the midset into an equilateral component $M_{\text{eq}}(x,y) = \{z: d(x,z) = d(y,z) = r\}$ and a far component $M_{\text{far}}(x,y) = \{z: d(x,z) = d(y,z) > r\}$, with $M(x,y) = M_{\text{eq}}(x,y) \sqcup M_{\text{far}}(x,y)$. Elements of M_{eq} form equilateral triangles with $\{x,y\}$; elements of M_{far} lie strictly outside $\bar{B}(x,r)$.

Remark 3.2. The decomposition $M(x,y) = M_{\text{eq}}(x,y) \sqcup M_{\text{far}}(x,y)$ is conceptually useful because it separates the two distinct geometric regimes in which equidistance occurs. The equilateral component consists of points at the same distance r from both x and y, while the far component consists of points strictly farther than r from both. This dichotomy is crucial because, as we show in Theorem 3.3, once a point is far enough from x, it is automatically equidistant from x and y-a phenomenon that drives many of the results that follow.

Theorem 3.3 (Automatic equidistance). If $d(x,z) > r$, then $d(y,z) = d(x,z)$, and $z \in M_{\text{far}}(x,y)$.

Proof. Applying the strong triangle inequality to $\{y,x,z\}: d(y,z) \leq \max\{d(y,x), d(x,z)\} = d(x,z)$ (since $d(x,z) > r = d(y,x)$). Applying it to $\{x,y,z\}: d(x,z) \leq \max\{d(x,y), d(y,z)\} = \max\{r, d(y,z)\}$. Since $d(x,z) > r$, this forces $d(y,z) \geq d(x,z)$. Together: $d(y,z) = d(x,z) > r$.

Remark 3.4. Theorem 3.3 is the key mechanism: it says that once a point is "far enough" from x (i.e., farther than y), it is automatically equidistant from x and y. This is a peculiarity of ultrametric geometry with no analogue in Euclidean or general metric spaces. In $\mathbb{R}^n$, the midset $M(x,y)$ is a hyperplane of codimension 1, and distant points are almost never equidistant.

Corollary 3.5. $M_{\text{far}}(x,y) = X \setminus \bar{B}(x,r)$.

Lemma 3.6.

$M_{\text{eq}}(x,y) = \bar{B}(x,r) \setminus (B(x,r) \cup B(y,r))$.
Proof. ($\subseteq$): If $d(x,z) = d(y,z) = r$, then $z \in \bar{B}(x,r)$ but $z \notin B(x,r)$ (since $d(x,z) = r \nless r$) and $z \notin B(y,r)$.

($\supseteq$) : Let $z \in \bar{B}(x,r) \setminus (B(x,r) \cup B(y,r))$. Then $d(x,z) \leq r$ and $d(x,z) \geq r$ (since $z \notin B(x,r)$), giving $d(x,z) = r$. For $d(y,z)$: since $z \in \bar{B}(x,r)$ and $y \in \bar{B}(x,r)$ (as $d(x,y) = r$), the center property gives $\bar{B}(x,r) = \bar{B}(y,r)$, so $d(y,z) \leq r$. But $z \notin B(y,r)$ gives $d(y,z) \geq r$. Hence $d(y,z) = r$.

Theorem 3.7 (Midset characterization). Let (X,d) be an ultrametric space with x, y distinct and $r = d(x,y)$. Then

$$M(x,y) = X \setminus (B(x,r) \cup B(y,r)).$$

In particular, $M(x,y)$ is a clopen subset of X.

Proof. By the decomposition $M = M_{\text{eq}} \sqcup M_{\text{far}}$, Lemma 3.6, and Corollary 3.5:

$$M(x,y) = [\bar{B}(x,r) \setminus (B(x,r) \cup B(y,r))] \sqcup [X \setminus \bar{B}(x,r)] = X \setminus (B(x,r) \cup B(y,r)).$$

The union $B(x,r) \cup B(y,r)$ is open (union of open sets) and closed (union of clopen sets by Proposition 2.1(iv)), so its complement is clopen.

Corollary 3.8. In any ultrametric space, the midset $M(x,y)$ is zero-dimensional (being clopen in a zero-dimensional space).

4. The Non-Empty Midset Property

Theorem 4.1. For a finite ultrametric space (X,d) with $|X| \geq 2$ and diameter partition classes $C_1, \dots, C_k$, the following are equivalent:

(a) $M(x,y) \neq \emptyset$ for all distinct $x,y \in X$ (NMP).
(b) $k = \kappa(X) \geq 3$.
(c) X contains three mutually equidistant points at distance $\operatorname{diam}(X)$.

Proof. (b) ⇔ (c): Representatives from three distinct diameter classes are pairwise at distance D.

(b) ⇒ (a): For distinct x,y with $r = d(x,y)$, we use Theorem 3.7: $M(x,y) = X \setminus (B(x,r) \cup B(y,r))$. If x,y are in the same class C_i, then $r < D$, and any z in a different class satisfies $d(x,z) = d(y,z) = D > r$, so $z \notin B(x,r) \cup B(y,r)$. If $x \in C_i, y \in C_j (i \neq j)$, then $r = D, B(x,D) = C_i, B(y,D) = C_j$; since $k \geq 3$, a third class provides the required element.

(a) ⇒ (b): If $k = 2$, pick $x \in C_1, y \in C_2$. Then $B(x,D) \cup B(y,D) = C_1 \cup C_2 = X$, so $M(x,y) = \emptyset$.

5. HaAr measure of midsets in $\mathbb{Z}_p$

We now deepen the p-adic analysis using the Haar measure on the compact group $\mathbb{Z}_p$. Let μ denote the normalized Haar measure with $\mu(\mathbb{Z}_p) = 1$. Recall that $\mu(a + p^k\mathbb{Z}_p) = p^{-k}$ for any $a \in \mathbb{Z}_p$ and $k \geq 0$.

Lemma 5.1. In $(\mathbb{Z}_p, |\cdot|_p)$, for $x \in \mathbb{Z}_p$ and $k \geq 0$:

(i) $B(x,p^{-k}) = x + p^{k+1}\mathbb{Z}_p$, with $\mu(B(x,p^{-k})) = p^{-(k+1)}$.
(ii) $\bar{B}(x,p^{-k}) = x + p^k\mathbb{Z}_p$, with $\mu(\bar{B}(x,p^{-k})) = p^{-k}$.
(iii) $\bar{B}(x,p^{-k})$ decomposes into p cosets of $p^{k+1}\mathbb{Z}_p$, each an open ball of radius p^{-k}.

Proof. (i) $z \in B(x,p^{-k})$ iff $|x-z|_p < p^{-k}$ iff $v_p(x-z) > k$ iff $x \equiv z (\operatorname{mod} p^{k+1})$. (ii) Similarly, $z \in \bar{B}(x,p^{-k})$ iff $v_p(x-z) \geq k$ iff $x \equiv z (\operatorname{mod} p^k)$. (iii) The coset $x + p^k\mathbb{Z}_p$ is the disjoint union $\bigsqcup_{j=0}^{p-1} (x + jp^k + p^{k+1}\mathbb{Z}_p)$.

Theorem 5.2. $\mathbb{Z}_p$ satisfies the NMP if and only if $p \geq 3$.

Proof. The diameter partition of $\mathbb{Z}_p$ (with $\operatorname{diam}(\mathbb{Z}_p) = 1$) consists of the p residue classes $\{i + p\mathbb{Z}_p\}_{i=0}^{p-1}$. By Theorem 4.1 (whose proof extends to compact ultrametric spaces with attained diameter), NMP holds iff $p \geq 3$.

Theorem 5.3 (Haar measure formula). Let $p \geq 3, x,y \in \mathbb{Z}_p$ distinct, $d(x,y) = |x-y|_p = p^{-k}$ for some $k \geq 0$. Then:

(i) $\mu(M_{\text{eq}}(x,y)) = (p-2) \cdot p^{-(k+1)}$,
(ii) $\mu(M_{\text{far}}(x,y)) = 1 - p^{-k}$,
(iii) $\mu(M(x,y)) = 1 - 2p^{-(k+1)}$.

Proof. By Lemma 5.1(iii), the closed ball $\bar{B}(x,p^{-k}) = x + p^k\mathbb{Z}_p$ decomposes into p cosets of $p^{k+1}\mathbb{Z}_p$. Two of these are $B(x,p^{-k})$ and $B(y,p^{-k})$ (distinct since $v_p(x-y) = k$, so $x \not\equiv y (\operatorname{mod} p^{k+1})$). By Lemma 3.6:

$$M_{\text{eq}}(x,y) = \bar{B}(x,p^{-k}) \setminus (B(x,p^{-k}) \cup B(y,p^{-k}))$$

consists of the remaining $p-2$ cosets, giving $\mu(M_{\text{eq}}) = (p-2) \cdot p^{-(k+1)}$. By Corollary 3.5, $M_{\text{far}}(x,y) = \mathbb{Z}_p \setminus \bar{B}(x,p^{-k})$, so $\mu(M_{\text{far}}) = 1 - p^{-k}$.

Combining: $\mu(M(x,y)) = (p-2)p^{-(k+1)} + 1 - p^{-k}$. Since $p^{-k} = p \cdot p^{-(k+1)}$:

$$\mu(M(x,y)) = (p-2)p^{-(k+1)} + 1 - p \cdot p^{-(k+1)} = 1 - 2p^{-(k+1)}.$$

Corollary 5.4. In $\mathbb{Z}_p (p \geq 3)$:

(i) $\inf_{x \neq y} \mu(M(x,y)) = 1 - 2/p = (p-2)/p$, attained by diameter pairs ($k = 0$).
(ii) $\mu(M(x,y)) \to 1$ as $d(x,y) \to 0$.
(iii) For $p = 3$: $\mu(M(x,y)) \in \{1/3, 7/9, 25/27, \dots\} = \{1 - 2 \cdot 3^{-(k+1)}\}_{k \geq 0}$.

Remark 5.5. The monotonicity in (ii) is geometrically natural: as x and y get closer ($k \to \infty$), their open balls $B(x,p^{-k})$ and $B(y,p^{-k})$ shrink (each has measure $p^{-(k+1)} \to 0$), so the midset $M(x,y) = \mathbb{Z}_p \setminus (B(x,p^{-k}) \cup B(y,p^{-k}))$ occupies nearly all of $\mathbb{Z}_p$. This contrasts sharply with the Euclidean case, where the midset (a hyperplane) has measure zero regardless of the distance between x and y.

Note also that the equilateral component M_{eq} is a union of $p-2$ scaled copies of $\mathbb{Z}_p$ (each coset $a + p^{k+1}\mathbb{Z}_p$ is isometric to $(\mathbb{Z}_p, p^{-(k+1)}|\cdot|_p)$), so $M_{\text{eq}}(x,y)$ inherits the self-similar structure of $\mathbb{Z}_p$.

Corollary 5.6. Let (X, d, μ) be a compact ultrametric space equipped with a Borel probability measure satisfying $\mu(B(x,r)) > 0$ for all x and $r > 0$. Then $\mu(M(x,y)) = 1 - \mu(B(x,r)) - \mu(B(y,r))$ where $r = d(x,y)$.

6. Recovery of the UMP result

Lemma 6.1. In an ultrametric space with the UMP, every midpoint is equilateral: if $\{m\} = M(x,y)$, then $d(x,m) = d(y,m) = d(x,y)$.

Proof. Set $r = d(x,y)$ and suppose $d(x,m) = d(y,m) = s > r$. By Corollary 3.5, every w with $d(x,w) > r$ belongs to $M(x,y)$. Since $|M(x,y)| = 1$, the only such point is m, giving $X = \bar{B}(x,r) \cup \{m\}$.

We show $M(x,m) = \emptyset$, contradicting UMP. For any $z \in \bar{B}(x,r)$: $d(x,z) \le r < s = d(x,m)$. By the isosceles inequality applied to $\{m,x,z\}$, since $d(m,x) = s \ne d(x,z)$, the two largest sides are equal: $d(m,z) = \max\{d(m,x), d(x,z)\} = s$. Hence $d(x,z) \le r < s = d(m,z)$ for all $z \in \bar{B}(x,r)$, so no element of $\bar{B}(x,r)$ is in $M(x,m)$. Also $m \notin M(x,m)$ since $d(x,m) = s \ne 0 = d(m,m)$. Thus $M(x,m) = \emptyset$.

Theorem 6.2. In an ultrametric space with the UMP, every closed ball containing at least two distinct points contains exactly three.

Proof. Let $a, b \in \bar{B}(x,r)$ be distinct, $s = d(a,b)$, and $\{m\} = M(a,b)$. By Lemma 6.1, $d(a,m) = d(b,m) = s$, so $\{a,b,m\}$ is equilateral and $m \in \bar{B}(a,s) \subseteq \bar{B}(x,r)$. By the UMP and equilateral symmetry: $M(a,m) = \{b\}$ and $M(b,m) = \{a\}$.

Suppose $z \in \bar{B}(x,r) \setminus \{a,b,m\}$. The UMP gives $z \notin M(a,b)$, so $d(a,z) \ne d(b,z)$; assume $d(a,z) < d(b,z)$. By the isosceles inequality in $\{a,b,z\}$: $d(b,z) = d(a,b) = s$. We check three exhaustive subcases for $\{a,m,z\}$:

If $d(a,z) < d(m,z)$: isosceles gives $d(m,z) = d(a,m) = s$, so $d(b,z) = d(m,z) = s$, hence $z \in M(b,m) = \{a\}$, contradicting $z \ne a$.

If $d(a,z) > d(m,z)$: isosceles gives $d(a,z) = d(a,m) = s$, so $d(a,z) = d(b,z) = s$, hence $z \in M(a,b) = \{m\}$, contradicting $z \ne m$.

If $d(a,z) = d(m,z)$: $z \in M(a,m) = \{b\}$, contradicting $z \ne b$.

Corollary 6.3 ([10]). Any ultrametric space with the UMP has at most three points.

Proof. Given distinct x, y with $\{m\} = M(x,y)$, Lemma 6.1 gives $d(x,m) = d(y,m) = r$. Any $w \notin \{x,y,m\}$ with $d(x,w) \le r$ contradicts Theorem 6.2; if $d(x,w) > r$, then $w \in M(x,y)$ (Theorem 3.3), giving $|M(x,y)| \ge 2$, a contradiction.

7. Dendrogram formula and equilateral rigidity

Theorem 7.1 (Dendrogram formula). Let (X, d) be a finite ultrametric space, $|X| = n$. For distinct x, y, let $v = \mathrm{lca}(x,y)$ with children $c_1, \dots, c_t$, with x descending from c_i and y from c_j. Then

$$|M(x,y)| = n - \lambda(c_i) - \lambda(c_j) \qquad (1)$$

Proof. The child c_i has leaf set $B(x, h(v))$ (the maximal open ball of radius $h(v) = d(x,y)$ containing x), so $|B(x,h(v))| = \lambda(c_i)$; similarly for c_j. Since $B(x,h(v))$ and $B(y,h(v))$ are disjoint (as x and y are in distinct children of v), Theorem 3.7 gives $|M(x,y)| = n - \lambda(c_i) - \lambda(c_j)$.

Remark 7.2. The formula (1) encodes a fundamental duality: the midset cardinality of a pair (x, y) is completely determined by the local tree structure at lca (x,y) (which branch contains x, which contains y) together with the global invariant n. In particular, two pairs with the same lca node and the same branch sizes produce identical midset cardinalities, even if the internal structure of those branches is entirely different.

Algorithmic note. Formula (1) provides a direct method for computing the full midset spectrum $\{|M(x,y)| : x \ne y\}$ from tree data: one need only traverse the internal nodes of the dendrogram and compute the two largest child leaf-counts $\lambda(c_i)$ and $\lambda(c_j)$, then subtract from n. This is useful in applications to hierarchical clustering and phylogenetic data.

Definition 7.3. For an internal node v with children $c_1, \dots, c_t$, the weight is $\sigma(v) = \lambda(c_{(1)}) + \lambda(c_{(2)})$, the sum of the two largest child leaf-counts.

Theorem 7.4 (Extremal values). For a finite ultrametric space with $n \ge 2$:

(i) $\max_{x \ne y} |M(x,y)| = n - 2$.
(ii) $\min_{x \ne y} |M(x,y)| = n - \max_v \sigma(v) = n - \sigma(\rho)$.

Proof.

(i) Each child contributes ≥ 1 leaf, so $|M| \le n - 2$. Equality: take a, b at minimum distance $r_{\min}$; then $B(a, r_{\min}) = \{a\}$ and $B(b, r_{\min}) = \{b\}$ (no point is closer to a than $r_{\min}$).
(ii) $\min |M| = n - \max_{v,i,j} \left(\lambda(c_i) + \lambda(c_j)\right) = n - \max_v \sigma(v)$. For non-root v: $\sigma(v) \le \lambda(v) \le n - 1$, since v 's leaves are a proper subset of X. For ρ : $\sigma(\rho)$ can reach n (when ρ has two children). Hence $\max_v \sigma(v) = \sigma(\rho)$.

Proposition 7.5. Only the root of $\mathcal{T}$ can cause NMP to fail: for every non-root node v, any pair merging at v has $|M| \ge 1$. NMP fails if and only if $\sigma(\rho) = n$, i.e., the root has exactly two children.

Theorem 7.6 (Equilateral rigidity). For a finite ultrametric space (X, d) with $n \geq 2$, the midset spectrum $\mathcal{S} = \{|M(x,y)|: x \neq y\}$ satisfies $|\mathcal{S}| = 1$ if and only if (X, d) is equilateral. When this holds, $|M(x,y)| = n-2$ for all pairs.

Proof. If all distances equal D, then $B(x,D) = \{x\}$ for all x, so $|M| = n-2$ for all pairs. Conversely, suppose $|M(x,y)| = k$ for all $x \neq y$. Let $r_{\min}$ be the minimum positive distance. Any pair (a,b) at distance $r_{\min}$ has $B(a, r_{\min}) = \{a\}$ and $B(b, r_{\min}) = \{b\}$, giving $k = n-2$.

Suppose for contradiction that $r_{\min} < D = \operatorname{diam}(X)$. Then a and b lie in the same diameter class C_i, so the open ball $B(a,D) \supseteq C_i \ni b$, giving $|B(a,D)| \geq 2$. Choose w from a different diameter class: $d(a,w) = D$, and

$$|M(a,w)| = n - |B(a,D)| - |B(w,D)| \leq n - 2 - 1 = n - 3 < k$$

a contradiction. Hence $r_{\min} = D$: all distances are equal.

Remark 7.7. The rigidity in Theorem 7.6 is specific to ultrametric spaces. In $\mathbb{R}^n$, many nonequilateral point configurations yield midsets of constant cardinality (in general position, every midset is a hyperplane, so "cardinality" is infinite). The ultrametric rigidity stems from the fact that ball sizes are discrete invariants tied to the distance values, creating an asymmetry between close and far pairs that only the equilateral configuration resolves.

8. Midset-convexity

We introduce a convexity notion adapted to midsets, and characterize it completely via the representing tree.

Definition 8.1. A subset $S \subseteq X$ of a metric space is midset-convex if $M(x,y) \cap S \neq \emptyset$ for all distinct $x, y \in S$.

By Theorem 3.7, S is midset-convex iff $S \nsubseteq B(x, d(x,y)) \cup B(y, d(x,y))$ for all distinct $x, y \in S$.

Lemma 8.2. Let (X, d) be a finite ultrametric space and $S \subseteq X$ with $|S| \geq 2$. Let $v^* = \operatorname{lca}(S)$ be the least common ancestor of S in $\mathcal{T}$. If w is a proper descendant of v^* and $x, y \in S$ both descend from w, then $M(x,y) \cap S \neq \emptyset$.

Proof. Since $v^* = \operatorname{lca}(S)$, the set S has elements in at least two distinct children of v^*. The node w, being a proper descendant of v^*, is contained in one child, say c_a, of v^*. Pick $z \in S$ descending from a different child $c_b (b \neq a)$. Then $d(x,z) \geq h(v^*) > h(w) \geq d(x,y)$, so Theorem 3.3 gives $d(y,z) = d(x,z)$, meaning $z \in M_{\text{far}}(x,y) \subseteq M(x,y)$. Hence $z \in M(x,y) \cap S$.

Lemma 8.2 shows that the only "dangerous" pairs are those whose least common ancestor is $v^* = \operatorname{lca}(S)$ itself. This leads to a sharp characterization.

Definition 8.3. The branching number of S at a node v is $\beta_S(v) = |\{c: c \text{ child of } v, S \cap \text{leaves}(c) \neq \emptyset\}|$.

Theorem 8.4 (Midset-convexity characterization). Let (X, d) be a finite ultrametric space and $S \subseteq X$ with $|S| \geq 2$. Then S is midset-convex if and only if $\beta_S(\operatorname{lca}(S)) \geq 3$.
Proof. Set $v^* = \operatorname{lca}(S)$ with children $c_1, \ldots, c_t$, and let $\beta = \beta_S(v^*)$.
($\Leftarrow$) : Assume $\beta \geq 3$. By Lemma 8.2, every pair with lca strictly below v^* satisfies the midset condition. For a pair (x,y) with lca $(x,y) = v^*: x \in c_i, y \in c_j (i \neq j)$, and $M(x,y) = X \setminus (\text{leaves}(c_i) \cup \text{leaves}(c_j))$ by Theorem 3.7. Since $\beta \geq 3$, there exists a child $c_\ell (\ell \neq i,j)$ with $S \cap \text{leaves}(c_\ell) \neq \emptyset$. Any element of this intersection lies in $M(x,y) \cap S$.
($\Rightarrow$) : If $\beta = 2$, say S meets only c_i and c_j, then $S \subseteq \text{leaves}(c_i) \cup \text{leaves}(c_j)$. Pick $x \in S \cap \text{leaves}(c_i)$, $y \in S \cap \text{leaves}(c_j)$. Then lca $(x,y) = v^*$, and $M(x,y) \cap S \subseteq S \setminus (\text{leaves}(c_i) \cup \text{leaves}(c_j)) = \emptyset$.

(Note: $\beta \geq 2$ always holds since $v^* = \operatorname{lca}(S)$ requires S to have elements in at least two children of v^*.)

Corollary 8.5. (X, d) has the NMP if and only if X itself is midset-convex.
Proof. $\operatorname{lca}(X) = \rho$ (the root), and $\beta_X(\rho) = \kappa(X)$ (the number of diameter classes). By Theorem 8.4, X is midset-convex iff $\kappa(X) \geq 3$, which is the NMP condition.

Corollary 8.6. Every subset of size ≥ 3 in an equilateral ultrametric space is midset-convex.
Proof. In an equilateral space, $\operatorname{lca}(S) = \rho$ and every point is a child of ρ, so $\beta_S(\rho) = |S| \geq 3$.
Corollary 8.7. A finite subset $S \subseteq \mathbb{Z}_p (p \geq 3)$ with $|S| \geq 2$ is midset-convex if and only if S intersects at least three residue classes modulo p^{k_0}, where $p^{-k_0} = \max\{d(x,y): x, y \in S\}$.

Example 8.8. In $\mathbb{Z}_3$, consider the following examples:

(i) The pair $S = \{0,1\}$ has $\operatorname{lca}(0,1) = \rho$ (the diameter), and these points lie in distinct residue classes modulo 3 (namely $0 + 3\mathbb{Z}_3$ and $1 + 3\mathbb{Z}_3$). Since $\beta_S(\rho) = 2 < 3$, the set S is not midset-convex. Indeed, $M(0,1) = \mathbb{Z}_3 \setminus (B(0,1) \cup B(1,1)) = \mathbb{Z}_3 \setminus ((0 + 9\mathbb{Z}_3) \cup (1 + 9\mathbb{Z}_3))$, which intersects the residue classes $\{2 + 9\mathbb{Z}_3, 3 + 9\mathbb{Z}_3, \ldots\}$, but $S \cap M(0,1) = \emptyset$ because both 0 and 1 belong to $B(0,1) \cup B(1,1)$

(ii) Adding any element z with $z \equiv 2 (\text{mod} 3)$ to form $S' = \{0,1,z\}$ gives $\beta_{S'}(\rho) = 3$, making S' midset-convex. (iii) The pair $T = \{0,3\}$ (with $d(0,3) = |0-3|_3 = 1/3$, i.e., $v_3(3) = 1$) has $\operatorname{lca}(0,3)$ at height $1/3$. Since $|T| = 2$ and $\beta_T(\operatorname{lca}(0,3)) = 2 < 3$, the set T is not midset-convex. One can verify: $M(0,3) \supseteq \{w: w \equiv 6 (\text{mod} 9)\}$, which is disjoint from $\{0,3\}$.

Remark 8.9. Theorem 8.4 reveals that midset-convexity is controlled by a single node-the least common ancestor of S-and a single integer-its branching number. The deeper nodes are irrelevant, thanks to the automatic equidistance phenomenon (Theorem 3.3): far-away elements always contribute to midsets, so only the "top-level" branching matters. This phenomenon has no Euclidean analogue: in $\mathbb{R}^n$, convexity depends on the global geometry of S, not on a single combinatorial invariant.

9. Concluding Remarks

The midset characterization $M(x,y) = X \setminus (B(x,r) \cup B(y,r))$ and the dendrogram formula $|M(x,y)| = n - \lambda(c_i) - \lambda(c_j)$ reduce midset questions in finite ultrametric spaces to rooted-tree combinatorics. The equilateral rigidity theorem shows that constant midset cardinality-an apparently mild condition-forces the entire distance structure to collapse. The midset-convexity characterization demonstrates that the tree structure is essential: only the top-level branching at the least common ancestor matters, a consequence of automatic equidistance. The Haar measure formula for $\mathbb{Z}_p$ illustrates how the general theory specializes to concrete, computable invariants.

We close with two questions.

Question 9.1. Characterize the NMP for bounded ultrametric spaces whose diameter is not attained. The equivalence $\sim_D$ then has a single class, and the NMP depends on finer structure.

Question 9.2. (Inverse spectrum problem.) Given $S \subset \mathbb{N}$, when does there exist a finite ultrametric space with midset spectrum S? Theorem 7.6 settles $|S| = 1$. A complete solution would connect to the combinatorics of integer partitions and rooted trees.

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Declaration of Competing Interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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